Uniqueness of the Partial-Wave Amplitudes*

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The following result is proved: Given all but a finite number of partial-wave amplitudes in a two-body scattering process, the remaining amplitudes are uniquely determined if either (a) the scattering amplitude has crossing symmetry or (b) there is an energy region in one of the crossed channels where the scattering is purely elastic. In case (a) the proof does not require a knowledge of the precise analytic structure of the scattering amplitude, while in case (b) the amplitude is assumed to satisfy the Mandelstam representation.

IMPORTANT insight into the structure of strong interactions of elementary particles has been achieved recently by the study of the ambiguities associated with the solutions of the Mandelstam representation. In this field, which was initiated by Froissart,[1] it is customary to assume[1,2] that the double-spectral functions associated with the process are given for some suitable values of the energy variables. Here we will consider a related problem where it will instead be assumed that all but a finite number N of partial waves associated with a two-body scattering amplitude are given, and the ambiguities associated with the remaining N partial waves will be examined. It will be proved that the remaining N partial waves are, in fact, uniquely determined except possibly for an additive s-wave constant if either (a) the scattering amplitude has crossing symmetry, or (b) in one of the crossed channels there exists an energy region where the scattering is purely elastic.[3] In case (a), a knowledge of the precise nature of the analyticity of the scattering amplitude is not required. It need only be assumed that the analytic continuation of the same amplitude determines the two processes we are interested in.

For simplicity, we will consider the scattering of two spinless particles of equal mass m. Let s denote the square of the center-of-mass energy in one of the three Mandelstam channels, and let θ be the corresponding scattering angle. The invariant momentum transfer t is defined through the usual relation

$$t=-2k^2(1-\cos\theta)\,, \qquad (1a)$$

i.e.,

$$\cos\theta=1+2t/(s-4m^2)\,, \qquad (1b)$$

where k denotes the center-of-mass momentum in this channel. The partial-wave amplitude $F_l(s)$ is defined as

$$F_l(s)=\left[\frac{s}{s-4m^2}\right]^{1/2}\frac{[\eta_l(s)e^{2i\delta_l(s)}-1]}{2i}\,, \qquad (2)$$

in a familiar notation.

To proceed further, let us assume that all the F_l's for l greater than some integer N are given. The proof goes through with trivial alterations when all but any finite number of the F_l's are given. The ambiguity we wish to resolve is associated with the possibility that for $l\leqslant N$, there may exist at least two amplitudes, say $F_l^{(1)}(s)$ and $F_l^{(2)}(s)$, which may describe the scattering for these l's. If this were possible, we may construct two scattering amplitudes $F^{(1)}(s,t)$ and $F^{(2)}(s,t)$ when $s\geqslant 4\mathrm{m}^2$ and $+1\leqslant\cos\theta\leqslant -1$ through the definitions

$$\begin{aligned}F^{(1)}(s,t)&=\sum_{l=0}^{N}(2l+1)F_l^{(1)}(s)\mathcal{P}_l\left(1+\frac{2t}{s-4m^2}\right)\\&\quad+\sum_{l=N+1}^{\infty}(2l+1)F_l(s)\mathcal{P}_l\left(1+\frac{2t}{s-4m^2}\right),\\F^{(2)}(s,t)&=\sum_{l=0}^{N}(2l+1)F_l^{(2)}(s)\mathcal{P}_l\left(1+\frac{2t}{s-4m^2}\right)\\&\quad+\sum_{l=N+1}^{\infty}(2l+1)F_l(s)\mathcal{P}_l\left(1+\frac{2t}{s-4m^2}\right).\end{aligned} \qquad (3)$$

Equation (3) shows that $F^{(1)}(s,t)-F^{(2)}(s,t)$ is a polynomial in t of degree N:

$$F^{(1)}(s,t)-F^{(2)}(s,t)=\sum_{n=0}^{N}b_n(s)t^n\,, \qquad (4)$$

where the b_n's are linear combinations of $F_l^{(1)}(s)-F_l^{(2)}(s)$ with coefficients which remain bounded as s tends to infinity.

Let us now examine case (a). Crossing symmetry implies that $F^{(1)}(s,t)=F^{(1)}(t,s)$, $F^{(2)}(s,t)=F^{(2)}(t,s)$, where we have assumed without loss of generality that the t channel is the one that is crossing symmetric to the s channel. Equation (4), therefore, gives

$$\sum_{n=0}^{N}b_n(s)t^n=\sum_{n=0}^{N}b_n(t)s^n\,, \qquad (5)$$

or $b_n(s)$ is a polynomial of degree N in s:

$$b_n(s)=\sum_{r=0}^{N}a_{n,r}s^r\,. \qquad (6)$$

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[1] M. Froissart, Phys. Rev. **123**, 1053 (1961).
[2] A. Martin, Phys. Rev. Letters **9**, 410 (1962).
[3] The result in case (b) has been independently obtained by A. Martin and by R. Oehme. The author is indebted to Professor Oehme for informing him of their results.

But the partial waves are bounded at infinity, and hence the b_n's must be constants for all n. (5) then shows that only $b_0(s)$ can be a nonzero constant. Thus, $F^{(1)}(s,t)$ and $F^{(2)}(s,t)$ can differ at most by a constant which is what we set out to prove. If we assume at this point that the η_0's go to zero as s tends to infinity, which seems a physically plausible assumption (dominance of inelastic processes at high energies), we see that this constant also must be zero.

To discuss case (b), we note that (4) implies that $F^{(1)}(s,t)-F^{(2)}(s,t)$ is regular for every finite t and fixed s. The absorptive part of $F^{(1)}(s,t)-F^{(2)}(s,t)$ in the t channel is, therefore, zero. If $\varphi_l^{(1)}(t)$ and $\varphi_l^{(2)}(t)$ denote the partial-wave amplitudes in the t channel, it follows that

$$\mathrm{Im}\,\varphi_l^{(1)}(t)=\mathrm{Im}\,\varphi_l^{(2)}(t) \tag{7}$$

for all l and $t\geqslant 4m^2$. By hypothesis, there is an interval $4m^2\leqslant t<t_1$ in the t channel where elastic unitarity is applicable. In this interval, as a consequence of (7), one finds

$$\mathrm{Re}\,\varphi_l^{(1)}(t)=\pm\mathrm{Re}\,\varphi_l^{(2)}(t). \tag{8}$$

Let M denote the number of subtractions in the Mandelstam representation. The case $\mathrm{Re}\,\varphi_l^{(1)}(t)=-\mathrm{Re}\,\varphi_l^{(2)}(t)$ may then be eliminated for $l>M$ by appealing to an elegant argument due to Martin.[2] Thus,

$$\varphi_l^{(1)}(t)=\varphi_l^{(2)}(t) \tag{9}$$

for $l>M$ and $4m^2\leqslant t<t_1$ and by analytic continuation, for all t. $F^{(1)}(s,t)-F^{(2)}(s,t)$ now reduces to a polynomial of degree M in s and the rest of the argument proceeds as before.

Let t_2 be the threshold for the second inelastic process in the t channel, and let the first inelastic channel be a two-body process too. A simple application of partial-wave unitarity then reveals that the partial waves in this new channel are also uniquely determined up to a phase for $t_1<t<t_2$. However, it seems difficult to carry this argument further since the phase may become complex for $t>t_2$ or $t<t_1$.

We may comment in conclusion on the relevance of the preceding remarks to the hypothesis that all particles are Regge poles.[4] The partial-wave amplitudes satisfy standard dispersion relations which, however, have a multiplicity of solutions.[5] It has been shown that for a class of processes, if these amplitudes are given for $l>N$, there is one preferred solution for $l\leqslant N$. This is equivalent to the statement that if we are given the interpolating partial-wave amplitude $F(s,l)$ of complex angular momentum theory,[6] (ignoring signature distinctions), which is *a priori* defined only for $\mathrm{Re}\,l$ greater than some integer M, the physical amplitudes for $l\leqslant M$ are, in fact, uniquely determined. The crucial point, then, is whether or not the analytic continuation of $F(s,l)$ for $\mathrm{Re}\,l\leqslant M$ coincides with the preferred solutions of the partial-wave dispersion relations.[7] It is important to observe in this connection that if we define an amplitude

$$F'(s,t)=\sum_{l=0}^{\infty}(2l+1)F(s,l)P_l(\cos\theta), \tag{10}$$

there is apparently no reason to believe that $F'(s,t)$ will satisfy either crossing symmetry or unitarity in one of the crossed channels.[8] Otherwise, it would be a simple matter to prove that all particles are Regge poles.

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[4] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **1**, 394 (1961); **8**, 41 (1962).
[5] L. Castillejo, R. H. Dalitz and F. J. Dyson, Phys. Rev. **101**, 453 (1956).
[6] M. Froissart, La Jolla Conference on the Theory of Weak and Strong Interactions, 1961 (unpublished); K. Bardakci, Phys. Rev. **127**, 1832 (1962).
[7] Since all the poles of $F(s,l)$ are Regge poles. See R. Oehme, Phys. Rev. Letters **9**, 358 (1962); Phys. Rev. **130**, 424 (1963).
[8] This point was emphasized to the author by Professor R. Oehme. Similar remarks have also been made by A. Martin (private communication to Professor R. Oehme).